Classical dynamics in deformed spaces

2003 J. Phys. A: Math. Gen. 36 1439

(http://iopscience.iop.org/0305-4470/36/5/317)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1439–1449 PII: S0305-4470(03)52882-7

Classical dynamics in deformed spaces

A Leznov[1,2,3] **and J Mostovoy**[4]

[1] Universidad Autónoma del Estado de Morelos, CCICAp, Cuernavaca, Morelos, Mexico
[2] Institute for High Energy Physics, Protvino, Moscow Region, 142280, Russia
[3] Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow Region, 141980, Russia
[4] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Av. Universidad s/n, CP 62210, Cuernavaca, Morelos, Mexico

E-mail: andrey@buzon.uaem.mx and jacob@matcuer.unam.mx

Received 22 October 2002, in final form 6 December 2002
Published 22 January 2003
Online at stacks.iop.org/JPhysA/36/1439

Abstract
We consider a 3-parametric linear deformation of the Poisson brackets in classical mechanics. This deformation can be thought of as the classical limit of dynamics in the so-called quantized spaces. Our main result is a description of the motion of a particle in the corresponding Kepler–Coulomb problem.

PACS numbers: 45.20.Jj, 02.40.Yy

1. Introduction

The set-up of a problem in quantum mechanics usually consists of specifying (a) the structure of the algebra of the observables and (b) the Hamiltonian of the problem. If the commutation relations in the algebra of the observables are deformations of the usual commutation relations between the coordinates and the momenta, the situation is often referred to as 'dynamics in a quantized space'. Schrödinger was apparently the first to consider such a situation: in [1] he calculated the energy levels for the Coulomb potential on a 3-sphere. For this problem the term 'quantized space' is rather appropriate as the spectrum of the distance operator on a 3-sphere is discrete even in the absence of any field.

One can obtain 'classical dynamics in a quantized space' by setting

$$\lim \frac{\mathrm{i}}{h}[A, B] = \{A, B\} \qquad \text{as} \quad h \to 0$$

where $\{\cdot, \cdot\}$ are the Poisson brackets of classical dynamics. We prefer to speak about 'classical dynamics in deformed spaces' (keeping in mind that it is not just the space but the whole dynamics being deformed).

In fact, one does not need a reference to the quantum theory in order to construct a deformation of the functional algebra of classical observables. However, we are specifically interested in the classical version of the 3-parametric family of deformations considered in [2–4] in the context of quantum mechanics.

This family arises naturally if one considers deformations of the commutation relations involving the four spacetime coordinates, ten generators of the Poincaré algebra (that is, infinitesimal translations and proper Lorentz transformations) and the unity element, with the property that the resulting algebra contains the Lorentz algebra as a subalgebra.

In what follows we will consider only three-dimensional non-relativistic problems. The usual commutation relations between the dynamic variables are replaced by the following set of brackets satisfying the Jacobi identity:

$$\begin{aligned}
&\{p_i, x_j\} = \delta_{ij} I - \frac{\epsilon_{ijk} l_k}{S} \qquad &&\{l_i, x_j\} = -\epsilon_{ijk} x_k \qquad &&\{I, x_i\} = \frac{x_i}{S} - \frac{p_i}{M^2} \\
&\{x_i, x_j\} = -\frac{\epsilon_{ijk} l_k}{M^2} &&\{l_i, p_j\} = -\epsilon_{ijk} p_k &&\{I, p_i\} = \frac{x_i}{L^2} - \frac{p_i}{S} \\
&\{p_i, p_j\} = -\frac{\epsilon_{ijk} l_k}{L^2} &&\{l_i, l_j\} = -\epsilon_{ijk} l_k &&\{I, l_i\} = 0.
\end{aligned} \tag{1}$$

Here x_i are the coordinates, and p_i and l_i are the components of the momentum and angular momentum, respectively; the function I is a deformation of the unity function. The three parameters L^2, M^2 and S have the dimensions of area, square of the momentum and action, respectively.

In general, these parameters may depend on rotation-invariant functions of the dynamic variables such as x^2, l^2, $(\boldsymbol{px})$ etc. In this case in the resulting nonlinear functional algebra the Jacobi identity are still satisfied. Here, however, we only consider the case of L^2, M^2 and S being constants. It should be stressed that we do not require the signs of L^2 and M^2 to be positive.

In order to obtain the flat space with the usual dynamics on the scale of the solar system, one has to assume that the values of L^2, M^2 and S are huge. In fact, taking M^2 and S to be infinite, we get the dynamics in the de Sitter world with curvature equal to $1/L^2$, so L is on the scale of the size of the Universe.

An important feature of relations (1) is their non-invariance with respect to time reversal. Namely, changing the signs of $\boldsymbol{p}$, $\boldsymbol{l}$ and of the brackets simultaneously one obtains relations of the same form but with the sign of the parameter S reversed.

Various particular cases of the above commutation relations are studied in the literature. The contraction giving the de Sitter world (that is $M^2, S = \infty$) has received most attention. The first work related to this case is due to Schrödinger [1] (who assumed the curvature to be positive, for the case of negative curvature see [5]). Many other works on the subject have appeared since; we only mention [6] which contains a discussion of the classical mechanics (and, in particular, the Kepler problem) in the space of constant curvature.

The case $L^2, S = \infty$ was first considered by Snyder [7] and the case $S = \infty$ is due to Yang [8]. The general case of the above commutation relations was introduced in [2], see also [4]. The quantum energy levels for the Kepler–Coulomb potential and the harmonic oscillator for this general case are calculated in [3].

Other approaches to deformed spaces exist (see, for example, [9]).

About the notation: we use bold italic letters for vectors. The scalar and vector products of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ will be denoted by $(\boldsymbol{ab})$ and $[\boldsymbol{ab}]$, respectively, the absolute value of $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$ and its square by a^2.

For simplicity, we always assume the particles to be of unit mass.

2. The canonical coordinates

2.1. The canonical coordinates

Depending on the signs of the constants L^2, M^2 and $L^2M^2 - S^2$, the functional algebra (1) is equivalent as a Lie algebra to one of the algebras $o(5)$, $o(1,4)$ or $o(2,3)$ [3]. According to the Darboux theorem, these algebras can be realized in terms of four pairs of independent canonically conjugate coordinates and momenta and two cyclic variables constructed from the elements of (1). In the algebraic language, the cyclic variables are the Casimir operators of the corresponding algebra.

The Casimir operators of the algebra under consideration are as follows:

$$K_2 = I^2 + \frac{x^2}{L^2} + \frac{p^2}{M^2} - \frac{2(\boldsymbol{px})}{S} - l^2\left(\frac{1}{S^2} - \frac{1}{L^2M^2}\right)$$

and

$$K_4 = (I\boldsymbol{l} - [\boldsymbol{xp}])^2 - \frac{(\boldsymbol{pl})^2}{M^2} + \frac{2(\boldsymbol{pl})(\boldsymbol{xl})}{S} - \frac{(\boldsymbol{xl})^2}{L^2}.$$

The Poisson brackets of K_2 and K_4 with any function vanish and their values can be fixed. In order to get a correct limit in the usual (that is, non-deformed) case we shall set $K_2 = 1$. As for the value of K_4, we will only consider the scalar representation of the algebra of the observables, that is, set

$$I\boldsymbol{l} = [\boldsymbol{xp}]. \tag{2}$$

This implies that $(\boldsymbol{pl}) = (\boldsymbol{xl}) = 0$ and, hence, that $K_4 = 0$. Relation (2) generalizes the definition of the angular momentum as a function of other variables to the case of the deformed space.

Imposing relation (2) we reduce the number of Darboux pairs by 1. An explicit realization of the algebra (1) in terms of three angular variables can be found in the appendix.

The equality $K_2 = 1$ (with l^2 expressed via $\boldsymbol{p}$, $\boldsymbol{x}$ and I) can be thought of as the equation defining the phase space of our dynamical system. In particular, if $M^2, S = \infty$ and $L^2 > 0$ this is the equation of $S^3 \times \mathbf{R}^3$ in $\mathbf{R}^7$ and relations (1) describe the dynamics on the 3-sphere of radius L. In fact, the phase space is a manifold whenever $S^2 = L^2M^2$. In general, however, it is not smooth: if $S^2 \neq L^2M^2$ there is a singularity at $I = 0$.

2.2. *Problems with spherical symmetry*

For problems with spherical symmetry (such as the Kepler problem considered below), it may be of use to consider the algebra of observables invariant under rotations. Namely, consider the algebra generated by the five spherically invariant variables: $r^2 \equiv x^2 + \frac{l^2}{M^2}$, $p'^2 \equiv p^2 + \frac{l^2}{L^2}$, $D \equiv (\boldsymbol{px}) + \frac{l^2}{S}$, I and l^2. The commutation relations of the nonlinear algebra generated by these five variables follow directly from (1):

$$\begin{aligned} &\{I, p'^2\} = 2\left(\frac{D}{L^2} - \frac{p'^2}{S}\right) && \{I, r^2\} = -\left(\frac{D}{M^2} - \frac{r^2}{S}\right) && \{I, D\} = \frac{r^2}{L^2} - \frac{p'^2}{M^2} \\ &\{p'^2, r^2\} = 4DI && \{D, p'^2\} = -2p'^2 I && \{D, r^2\} = 2r^2 I. \end{aligned} \tag{3}$$

The nonlinear functional algebra (3) is generated by five elements and has three cyclic variables: two of them, namely K_2 and K_4, come from the initial algebra. The third cyclic variable is l^2 which commutes with all the generators of (3). Thus, by the Darboux theorem, this algebra can be realized in terms of one coordinate and one momentum. We choose r as the coordinate. (In the non-deformed case this is just the radial variable.)

The values of K_2 and K_4 being fixed, the quantities D, p' and I depend only on r, the corresponding canonically conjugate momentum which we denote by ρ, and l^2. Taking the squares of the vectors on both sides of (2) and using the fact that $K_2 = 1$ after some manipulations we obtain

$$p'^2 = \frac{1}{r^2}(D^2 + l^2). \tag{4}$$

From the last equality of (3) we get that

$$I = \frac{1}{r} D_\rho.$$

Substituting this expression together with (4) into the equality $K_2 = 1$, we obtain an equation on D

$$1 = \frac{D_\rho^2}{r^2} + \frac{D^2 + l^2}{M^2 r^2} + \frac{r^2}{L^2} - \frac{2D}{S} + l^2 \left(\frac{1}{S^2} - \frac{1}{L^2 M^2} \right)$$

with the general solution

$$D = \frac{M^2 r^2}{S} - \sqrt{\left(1 + r^2 M^2 \left(\frac{1}{S^2} - \frac{1}{L^2 M^2} \right) \right) (M^2 r^2 - l^2)} \cos \frac{\rho}{M}. \tag{5}$$

As a corollary we have

$$I = \sqrt{\left(1 + r^2 M^2 \left(\frac{1}{S^2} - \frac{1}{L^2 M^2} \right) \right) \left(1 - \frac{l^2}{M^2 r^2} \right)} \sin \frac{\rho}{M}. \tag{6}$$

Thus all elements of the nonlinear functional algebra given by (3) are expressed in terms of the canonically conjugate pair (r, ρ) and the square of the angular momentum l^2.

3. Free motion

The Hamiltonian of the free motion is chosen so as to commute with infinitesimal translations and rotations, that is with $\boldsymbol{p}$ and $\boldsymbol{l}$. The simplest such expression which gives the correct limit as the parameters tend to infinity is

$$E = \frac{1}{2} \left(p^2 + \frac{l^2}{L^2} \right).$$

It follows immediately from the commutation relations that the vectors $\boldsymbol{p}$ and $\boldsymbol{l}$ are constant. Also,

$$\dot{\boldsymbol{x}} = I \boldsymbol{p} - \frac{[\boldsymbol{x}\boldsymbol{l}]}{L^2} + \frac{[\boldsymbol{p}\boldsymbol{l}]}{S}$$

and

$$\dot{I} = \frac{p^2}{S} - \frac{(\boldsymbol{p}\boldsymbol{x})}{L^2}.$$

From this

$$\ddot{I} + \frac{2E}{L^2} I = 0$$

and, hence,

$$I = I_0 \cos \left(\frac{\sqrt{2E}}{L} (t - t_0) \right).$$

Now, taking the vector product of (2) with $\boldsymbol{p}$ one has

$$(\boldsymbol{p}\boldsymbol{x})\boldsymbol{p} - p^2 \boldsymbol{x} = I [\boldsymbol{l}\boldsymbol{p}]$$

from where

$$\boldsymbol{x} = \left(\frac{L^2}{S} + \frac{I_0}{p^2} L \sqrt{2E} \sin \left(\frac{\sqrt{2E}}{L} (t - t_0) \right) \right) \boldsymbol{p} - \frac{I_0}{p^2} \cos \left(\frac{\sqrt{2E}}{L} (t - t_0) \right) [\boldsymbol{l}\boldsymbol{p}]. \tag{7}$$

The value of I_0 can be found from the condition $K_2 = 1$:

$$I_0 = \frac{p}{\sqrt{2E}}\sqrt{1 + 2E\left(\frac{L^2}{S^2} - \frac{1}{M^2}\right)}.$$

In the case $L^2 = \infty$ the expression for $\boldsymbol{x}$ should be modified to read

$$\boldsymbol{x} = \boldsymbol{x}_0 + \left(I_0\boldsymbol{p} + \frac{[\boldsymbol{pl}]}{S}\right)t + \frac{p^2}{2S}\boldsymbol{p}t^2$$

with

$$I_0 = \sqrt{1 - \frac{p^2}{M^2} + \frac{l^2}{S^2} + \frac{2(\boldsymbol{px}_0)}{S}}.$$

The orbits of the free motion are elliptic, parabolic or hyperbolic according to whether L^2 is positive, zero or negative, respectively. This should not be surprising given that L^{-2} can be interpreted as the curvature of the space. For example, set M^2, $S = \infty$ and take $L^2 > 0$. Then the elliptic orbits (7) can be obtained from the geodesics on the 3-sphere

$$1 = I^2 + \frac{x^2}{L^2}$$

by projecting them orthogonally on a Euclidean 3-space. In particular, the farthest distance from the origin reached by a freely moving particle is the same for all orbits and equals L.

4. The Kepler problem

4.1. The equation of the orbit

The Hamiltonian for the Kepler problem is chosen (as in [3]) so as to commute with the Runge–Lenz vector $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = [\boldsymbol{pl}] + \frac{\alpha\boldsymbol{x}}{r} = \frac{1}{I}(p^2\boldsymbol{x} - (\boldsymbol{px})\boldsymbol{p}) + \frac{\alpha\boldsymbol{x}}{r} \tag{8}$$

where r is given by $r^2 = x^2 + \frac{l^2}{M^2}$.

The commutation relations for the components of $\boldsymbol{A}$ are as follows:

$$\{A_i, A_j\} = 2\left(E - \frac{l^2}{L^2}\right)\epsilon_{ijk}l_k$$

where E stands for the expression

$$E = \frac{1}{2}\left(p^2 + \frac{l^2}{L^2} + \frac{2\alpha I}{r} - \frac{\alpha^2}{M^2r^2}\right). \tag{9}$$

Also,

$$\{l_i, A_j\} = -\epsilon_{ijk}A_k$$

and, hence,

$$\{l^2, A_j\} = 2\epsilon_{ijk}l_jA_k.$$

To see that E commutes with all of the components of $\boldsymbol{A}$ first note that

$$A^2 = \alpha^2 + l^2\left(2E - \frac{l^2}{L^2}\right). \tag{10}$$

Now,

$$2\{E, A_i\} = \left\{\frac{A^2}{l^2}, A_i\right\} - \alpha^2\left\{\frac{1}{l^2}, A_i\right\} + \frac{1}{L^2}\{l^2, A_i\} = 0.$$

Thus E can be chosen to be the Hamiltonian for the Kepler problem.

The three vectors $\boldsymbol{x}, \boldsymbol{p}$ and $\boldsymbol{A}$ lie in the plane orthogonal to the vector of angular momentum. Let us choose the polar coordinates $(|\boldsymbol{x}|, \phi)$ in this plane with the direction $\phi = 0$ being given by the Runge–Lenz vector. (If $\boldsymbol{A} = 0$ we take $\phi = 0$ to be an arbitrary ray passing through the origin.)

Taking the scalar product of (8) with $\boldsymbol{x}$ one has

$$\frac{A|\boldsymbol{x}|\cos\phi - \alpha r}{l^2} = I - \frac{\alpha}{M^2 r}. \tag{11}$$

Taking the vector product of $\boldsymbol{A}$ and $\boldsymbol{l}$ we have

$$[\boldsymbol{A}\boldsymbol{l}] = -l^2\boldsymbol{p} + \frac{\alpha}{r}[\boldsymbol{x}\boldsymbol{l}]. \tag{12}$$

The scalar product of (12) with $\boldsymbol{x}$ gives

$$(\boldsymbol{p}\boldsymbol{x}) = -\frac{1}{l^2}([\boldsymbol{A}\boldsymbol{l}], \boldsymbol{x}) = \frac{A|\boldsymbol{x}|\sin\phi}{l}. \tag{13}$$

Now the orbits can be found as follows. In the expression $K_2 = 1$ one can substitute $(\boldsymbol{p}\boldsymbol{x})$ from (13), I from (11) and p^2 from (9) and (10). The resulting equation of the orbit is as follows:

$$1 + \frac{2A|\boldsymbol{x}|\sin\phi}{lS} = \frac{(A|\boldsymbol{x}|\cos\phi - \alpha r)^2}{l^4} + \frac{r^2}{L^2} + \frac{A^2 - \alpha^2}{l^2 M^2} - \frac{l^2}{S^2}. \tag{14}$$

In the Cartesian coordinates $(x, y) = (|\boldsymbol{x}|\cos\phi, |\boldsymbol{x}|\sin\phi)$ equation (14) takes the following form:

$$x^2\left(\frac{A^2+\alpha^2}{l^4} + \frac{1}{L^2}\right) + y^2\left(\frac{\alpha^2}{l^4} + \frac{1}{L^2}\right) - \frac{2Ay}{lS} + \frac{A^2}{l^2M^2} - l^2\left(\frac{1}{S^2} - \frac{1}{L^2M^2}\right) - 1$$
$$= \frac{2\alpha A}{l^4} x\sqrt{x^2 + y^2 + \frac{l^2}{M^2}}. \tag{15}$$

This equation defines a branch of a plane quartic. The other branch of this quartic describes the orbit with the same values of A and $\boldsymbol{l}$ but with the opposite value of α. Note that changing $\boldsymbol{l}$ for $-\boldsymbol{l}$ we obtain, generally, a different orbit. (Recall that the commutation relations (1) are not invariant under time reversal.)

Interestingly, it was Giovanni Domenico Cassini (1625–1712) who first suggested that the trajectories of planets are quartic curves. These curves, now known as Cassinian ovals, are rather different from the trajectories (15) obtained above. A Cassinian oval is defined as the locus of points with the product of their distances from two fixed points equal to a fixed constant.

4.2. The parametric form for the orbit

From now on we will assume for simplicity that L^2 is positive. Taking into account the fact that $K_2 = 1$, this implies, in particular, that all orbits are closed.

The curve (15) can be parametrized as follows. The condition $L^2 > 0$ means that we can choose a, b so that

$$a^2 + b^2 = \frac{\alpha^2 + A^2}{l^4} + \frac{1}{L^2}$$

and

$$ab = \frac{A\alpha}{l^4}.$$

Explicitly,

$$a^2, b^2 = \frac{1}{2}\left(\frac{\alpha^2 + A^2}{l^4} + \frac{1}{L^2} \pm \sqrt{\left(\frac{\alpha^2 + A^2}{l^4} + \frac{1}{L^2}\right)^2 - \frac{4A^2\alpha^2}{l^8}}\right)$$

and we take $a^2 > b^2$.

Equation (15) can be written as

$$(ax - br)^2 + \left(\nu y - \frac{A}{lS\nu}\right)^2 = \gamma^2.$$

Here

$$\nu^2 = \frac{1}{2}\left(\frac{\alpha^2 - A^2}{l^4} + \frac{1}{L^2} + \sqrt{\left(\frac{\alpha^2 + A^2}{l^4} + \frac{1}{L^2}\right)^2 - \frac{4A^2\alpha^2}{l^4}}\right)$$

and

$$\gamma^2 = 1 + l^2\left(\frac{1}{S^2} - \frac{1}{L^2M^2}\right)\frac{a^2}{\nu^2}.$$

In the non-deformed limit $\gamma = 1$ and

$$\nu^2 = \frac{\alpha^2 - A^2}{l^4} = -\frac{2E}{l^2}$$

for $E < 0$ and $\nu^2 = 0$ otherwise.

We can set

$$ax - br = \gamma\cos\theta \qquad \nu y - \frac{A}{lS\nu} = \gamma\sin\theta$$

so that

$$\begin{aligned}
r &= \frac{1}{a^2 - b^2}\left(b\gamma\cos\theta + |a|\sqrt{\gamma^2\cos^2\theta + (a^2 - b^2)\left(\left(\frac{\gamma}{\nu}\sin\theta + \frac{A}{lS\nu^2}\right)^2 + \frac{l^2}{M^2}\right)}\right)\\
x &= \frac{1}{a^2 - b^2}\left(a\gamma\cos\theta + b\frac{|a|}{a}\sqrt{\gamma^2\cos^2\theta + (a^2 - b^2)\left(\left(\frac{\gamma}{\nu}\sin\theta + \frac{A}{lS\nu^2}\right)^2 + \frac{l^2}{M^2}\right)}\right) \qquad (16)\\
y &= \frac{\gamma}{\nu}\sin\theta + \frac{A}{lS\nu^2}.
\end{aligned}$$

In the classical limit the above parametrization gives the usual trigonometric parametrization of the elliptic trajectory. There exists also a generalization of the hyperbolic parametrization. Write (15) as

$$(bx - ar)^2 - \left(\mu y + \frac{A}{lS\mu}\right)^2 = \delta^2$$

with

$$\mu^2 = -\frac{1}{2}\left(\frac{\alpha^2 - A^2}{l^4} + \frac{1}{L^2} - \sqrt{\left(\frac{\alpha^2 + A^2}{l^4} + \frac{1}{L^2}\right)^2 - \frac{4A^2\alpha^2}{l^4}}\right) \qquad (17)$$

and

$$\delta^2 = 1 - l^2\left(\frac{1}{S^2} - \frac{1}{L^2M^2}\right)\frac{b^2}{\mu^2}. \tag{18}$$

Assume that $\delta^2 > 0$. Then setting

$$bx - ar = \delta\cosh\theta \qquad \mu y + \frac{A}{lS\mu} = \delta\sinh\theta$$

we obtain the hyperbolic parametrization of the orbit. (The case $\delta^2 < 0$ is entirely similar.)

Note that $\mu^2\nu^2 = \frac{A^2}{l^4L^2}$ so in the non-deformed limit only one of the parametrizations is meaningful for the given values of α and A, the other just giving the equation of the orbit. In the deformed case both parametrizations can be meaningful for the same closed orbit. In this case the hyperbolic parametrization is two-valued, and the parameter θ belongs to a closed interval.

In the non-deformed limit the hyperbolic parametrization remains two-valued with the parameter θ varying over all real numbers. The two branches correspond to the two possible values of $I = \pm 1$. One branch gives the usual parametrization of a hyperbolic trajectory. The second branch parametrizes a trajectory with the same value of A and l but with the opposite sign of α.

4.3. Dependence on time

Taking the Poisson bracket of $\boldsymbol{x}$ with the Hamiltonian we obtain the equations of motion:

$$\dot{\boldsymbol{x}} = I\boldsymbol{p} - \frac{[\boldsymbol{xl}]}{L^2} + \frac{[\boldsymbol{pl}]}{S} + \frac{\alpha\boldsymbol{x}}{Sr} - \frac{\alpha\boldsymbol{p}}{M^2r}.$$

Substituting I and $\boldsymbol{p}$ from (11) and (12) respectively, we get

$$\dot{\boldsymbol{x}} = \frac{1}{l^4}(Ax - \alpha r)\left(\frac{\alpha}{r}[\boldsymbol{xl}] - [\boldsymbol{Al}]\right) - \frac{[\boldsymbol{xl}]}{L^2} + \frac{\boldsymbol{A}}{S}.$$

In particular, for the coordinate y we have

$$\dot{y} = \frac{1}{l^3}\left(\left(A^2 + \alpha^2 + \frac{l^4}{L^2}\right)x - \alpha A\left(\frac{x^2}{r} + r\right)\right)$$

or, equivalently,

$$\dot{y} = -\frac{l}{r}(ax - br)(bx - ar).$$

It follows from (16) that

$$\frac{r}{bx - ar} = -\frac{|a|}{a}\left(\frac{b\gamma\cos\theta}{\sqrt{\Delta}} + |a|\right)\frac{1}{a^2 - b^2}$$

where Δ is the expression under the square root in (16). Thus

$$\frac{|a|}{a}l\nu(a^2 - b^2)\,\mathrm{d}t = \mathrm{d}\theta\left(|a| + \frac{b\gamma\cos\theta}{\sqrt{\Delta}}\right). \tag{19}$$

The function Δ only depends on $\sin\theta$ so the period of motion can be found without integrating (19)

$$T = \frac{2\pi|a|}{l\nu(a^2 - b^2)}.$$

Note that for A, l and α fixed, the period only depends on L^2 and not on M^2 and S.

One can rewrite (19) as

$$\frac{|a|}{a}(a^2-b^2)l\mathrm{d}t = \frac{|a|}{\nu}\mathrm{d}\theta + \frac{b}{\mu}\frac{\mathrm{d}q}{\sqrt{q^2+\delta^2}}$$

where

$$q = \frac{\mu}{\nu}\gamma\sin\theta + \frac{(a^2-b^2)lL}{\nu S}$$

δ^2 is as in (18) and μ is defined as in (17) so that $a^2-b^2=\mu^2+\nu^2$. Integrating, one gets the following parametrization of the time t by θ:

$$\frac{|a|}{a}(t-t_0)(a^2-b^2)l = \frac{|a|}{\nu}\theta + \frac{b}{\mu}\sinh^{-1}\left(\frac{\mu\gamma}{\nu\delta}\sin\theta + \frac{(a^2-b^2)lL}{\delta\nu S}\right). \qquad (20)$$

(The above answer is valid if $\delta^2>0$; obvious modifications are otherwise needed in (20).)

The standard trigonometric parametrization of t by θ in the non-deformed Kepler problem can be obtained from (20) by setting first S and M^2 to be infinite and then taking the limit as L^2 tends to infinity. (Note that as L^2 tends to ∞, the parameter μ tends to 0, as $\mu=\frac{A}{\nu lL}$.)

5. The Hamilton–Jacobi equation

The spherically symmetric part of the Hamilton–Jacobi equation is obtained by substituting (4), (5) and (6) into the explicit expression for the Hamiltonian function

$$2E = \frac{1}{r^2}(D^2+l^2) + \frac{2\alpha}{r}I - \frac{\alpha^2}{M^2r^2}.$$

Namely, we obtain

$$2Er^2 - l^2 + \frac{\alpha^2}{M^2} = \left(\frac{M^2r^2}{S} - \sqrt{\left(1+r^2M^2\left(\frac{1}{S^2}-\frac{1}{L^2M^2}\right)\right)(M^2r^2-l^2)}\cos\frac{W_r}{M}\right)^2$$
$$+\frac{2\alpha}{M}\sqrt{\left(1+r^2M^2\left(\frac{1}{S^2}-\frac{1}{L^2M^2}\right)\right)(M^2r^2-l^2)}\sin\frac{W_r}{M}$$

where W is the radial part of the action function. One arrives at the same expression considering the quasi-classical limit for the Schrödinger equation in a deformed (quantized) space.

We do not know a direct method of solving this (rather unusual at first sight) equation. In fact, one only needs to know the second derivative W_{rE} in order to describe r as a function of time: $r(t)$ can be found from the equality

$$t-t_0 = \int \mathrm{d}r\, W_{rE}.$$

Below we obtain an algebraic equation for the second derivative W_{rE} of the radial part of the action.

Let us introduce the following notation:

$$u = \frac{M^2r^2}{S} - \sqrt{\left(1+r^2M^2\left(\frac{1}{S^2}-\frac{1}{L^2M^2}\right)\right)(M^2r^2-l^2)}\cos\frac{W_r}{M}$$

$$v = \sqrt{\left(1+r^2M^2\left(\frac{1}{S^2}-\frac{1}{L^2M^2}\right)\right)(M^2r^2-l^2)}\sin\frac{W_r}{M} - \frac{\alpha}{M}.$$

In this notation the Hamilton–Jacobi equation becomes

$$u^2 + \frac{2\alpha}{M} v = 2Er^2 - l^2 - \frac{\alpha^2}{M^2} \equiv \theta. \tag{21}$$

Taking the derivative of both sides with respect to E we have

$$uv = Mr^2 \left(\frac{1}{W_{rE}} - \frac{\alpha}{S} \right) \equiv F. \tag{22}$$

Finally, taking into account that $\sin^2 x + \cos^2 x = 1$ we obtain

$$v^2 - \frac{2M^2r^2}{S} u = M^2r^2 \left(1 - \frac{2E}{M^2} - \frac{r^2}{L^2} - l^2 \left(\frac{1}{S^2} - \frac{1}{M^2L^2} \right) \right) \equiv \bar{\theta}. \tag{23}$$

The condition of self-consistency of the above three equations with respect to only two functions u, v leads to an equation for the function F in the form

$$(\overline{F}^2 - \theta\bar{\theta})^2 = \overline{F}(\overline{F}^2 - 3\theta\bar{\theta}) + (s^2\bar{\theta}^3 + w^2\theta^3)\overline{F}$$

where $s = \frac{\alpha}{M}$, $w = \frac{M^2r^2}{S}$ and $\overline{F} \equiv F + 4sw$.

6. Conclusions

The main result of this paper is a parametric solution of the Kepler problem in a deformed space whose construction involves three parameters with dimensions of length, momentum and action, respectively.

The orbits of particles in this version of the Kepler problem turn out to be (branches of) quartic curves. This should be compared with the analysis of Higgs [6] who studies the orbits in the case where there is only one non-trivial deformation parameter, namely the length L. In Higgs' picture the orbits are elliptic, parabolic or hyperbolic just as in the non-deformed case. This, however, is due to a special choice of coordinates used in [6]. These coordinates, though well suited to the geometry of the problem, have no physical significance. The coordinates we use are distinguished among all possible coordinate systems as 'the physical coordinates'. The transformation between Higgs' coordinates and the physical coordinates is very simple: in our notation, Higgs' coordinates can be written as x_i/I.

The fact that the Kepler problem can be solved in the deformed space gives some hope that other problems (for example, construction of field theories) could also have reasonable solutions in deformed spaces. The discussion of this subject is, however, beyond the scope of this paper.

Acknowledgments

We would like to thank Carlos Villegas-Blas for useful advice. The second author was supported by the grant J-33616E awarded by CONACyT.

Appendix. Structure of the algebra (1)

Throughout this appendix we assume that $M^2 > 0$ and

$$\frac{1}{S^2} - \frac{1}{L^2M^2} > 0.$$

Take the spherical coordinates with the angles (ϕ, θ, χ) in $\mathbf{R^4}$. Let $(\boldsymbol{u}, u_4)$ be the unit vector

$$(\boldsymbol{u}, u_4) = (\sin\chi \sin\theta \sin\phi, \sin\chi \sin\theta \cos\phi, \sin\chi \cos\theta, \cos\chi)$$

and set

$$p_\chi = \frac{\partial}{\partial\chi} \qquad p_\theta = \frac{\partial}{\partial\theta} \qquad p_\phi = \frac{\partial}{\partial\phi}.$$

Define the vectors $\boldsymbol{F}$ and $\boldsymbol{L}$ by

$$\begin{aligned} F_1 &= \sin\theta\sin\phi\frac{\partial}{\partial\chi} + \cot\chi\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cot\chi\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \\ F_2 &= \sin\theta\cos\phi\frac{\partial}{\partial\chi} + \cot\chi\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \cot\chi\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \\ F_3 &= \cos\theta\frac{\partial}{\partial\chi} + \cot\chi\sin\theta\frac{\partial}{\partial\theta} \end{aligned}$$

and

$$\begin{aligned} L_1 &= \sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi} \\ L_2 &= \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi} \\ L_3 &= -\frac{\partial}{\partial\phi} \end{aligned}$$

respectively.

Now,

$$\begin{aligned} I &= (\boldsymbol{u}\boldsymbol{F}) + u_4\rho \\ \frac{1}{M}\left(\boldsymbol{p} - \frac{M^2}{S}\boldsymbol{x}\right) &= \rho\boldsymbol{u} + [\boldsymbol{L}\boldsymbol{u}] - u_4\boldsymbol{F} \\ \boldsymbol{x}M\sqrt{\frac{1}{S^2} - \frac{1}{L^2M^2}} &= \boldsymbol{F} \\ \boldsymbol{l}\sqrt{\frac{1}{S^2} - \frac{1}{L^2M^2}} &= \boldsymbol{L}. \end{aligned} \tag{24}$$

Here ρ is a constant.

References

[1] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9
[2] Leznov A N and Khruschev V V 1973 *IHEP Preprint* no. 73-38 Serpukhov
[3] Leznov A N 2002 *Nucl. Phys.* B **640** 469
[4] Khruschev V V and Leznov A N 2002 *Preprint* hep-th/0207082
[5] Infeld L and Schild A 1945 *Phys. Rev.* **67** 121
[6] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
[7] Snyder H 1947 *Phys. Rev.* **71** 38
[8] Yang C N 1947 *Phys. Rev.* **72** 874
[9] Curtright T L and Zachos C K 2002 *New J. Phys.* **4** 83